Final Exam

MTH 516/616, Semester 2, 2021-22

Duration: 3 hours

Maximum Points: 60

Directions: Please ensure that you sufficiently explain and justify all intermediate arguments leading to any conclusions that you may draw along the way. Each statement (or argument) in your solution must be clearly explained, and must be devoid of any logical gaps or inconsistencies.

1. Let M and N be connected n-manifolds with fundamental classes $[M]$ and $[N]$, respectively. For a continuous map $f : M \to N$, the *degree of* f is an integer d such that $f([M]) = d([N])$. [10+10+10]

   (a) Show that there is a map $f : M \to S^n$ of degree 1.

   (b) Show that a p-sheeted covering projection $p : M \to N$ has degree $\pm p$.

   (c) Show that for a degree 1 map $f : M \to N$, the induced homomorphisms at the level of fundamental group and homology are both surjective.

2. Let X be a connected space with a basepoint x_0. [15+5]

   (i) For $i \geq 1$, show that the following statements are equivalent.

      (a) Every map $S^i \to X$ is homotopic to a constant map.

      (b) Every map $S^i \to X$ extends to a map $D^{i+1} \to X$.

      (c) $\pi_i(X, x_0) = 0$.

   (ii) Show that an n-connected, n-dimensional CW-complex is contractible.

3. If M is a connected compact orientable n-manifold, a homeomorphism $f : M \to M$ is *orientation-preserving* if f_* takes the fundamental class to itself, and *orientation reversing*, otherwise. Use the cup product structure to show that every homeomorphism $f : \mathbb{C}P^2 \to \mathbb{C}P^2$ is orientation-preserving. [10]